CONSTRUCTION OF SECOND ORDER ROTATABLE DESIGNS THROUGH A PAIR OF BALANCED INCOMPLETE BLOCK DESIGNS

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SUMMARY

A new unified method of constructing Second Order Rotatable designs using two suitably chosen Balanced Incomplete Block Designs is suggested. The method of Central Composite Rotatable Design, Box and Behnken [2] method, Das and Narasimham [6] method are shown to be particular cases of this new unified method.

INTRODUCTION

Rotatable Designs were introduced by Box and Hunter [3] for the exploration of Response Surfaces. Box and Hunter [3], Bose and Draper [1]. Box and Behnken [2], Draper [8], Das [4], Das and Narasimham [6], Das [5] Raghavarao [11], Tyagi [13], Saha and Das [12], Dey and Kulshreshtha [7], Gupta and Dey [9] and Nigam [10] suggested different methods for construction of Second Order Rotatable Designs.

In particular, Saha and Das [12] gave a particular unified approach for constructing four level Second Order Rotatable Designs using partially balanced arrays of strength two. In this method, they replace the 1's (ones) and O's (Zeros) in the incidence matrix by the two symbols α and β and use multiplication in Das and Narasimham [6] sense. Because of two non-zero symbols in each block these designs have naturally a large number of points. Dey

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and Kulshreshtha [7] used a similar approach for six level Second Order Rotatable Designs. Further Gupta and Dey [9] and Nigam [10] studied this particular type of unified approach for construction of four and six level Second order Rotatable Designs in greater detail.

In this paper, a general unified method for the construction of Second Order Rotatable Designs using two Balanced Incomplete Block Designs, without any further set of points is suggested.

In this new approach we use two Balanced Incomplete Block Designs, one with symbols α and 'o' (zero) and another with symbols 'β' and 'o' use appropriate multiplications in Das and Narasimham [6] sense as usual to generate the design points for Second Order Rotatable Design. Because of the presence of several zeros in each block the number of design points will be comparatively less in the method. Further these designs contain five or three levels.

It is established that the method of Central Composite Design, Box and Behnken [2] method and Das and Narasimham [6] method are particular cases of this new unified approach.

Methods of Constructing Second Order Rotatable Designs Using a Balanced Incomplete Block Design

Box and Behnken [2] gave a method for the construction of Second Order Rotatable Design using a Balanced Incomplete Block Designs with $r=3\lambda$, where 'λ' is the pairwise replication constant and 'r' is the number of times that a treatment is replicated.

Das and Narasimham [6] gave a general method of constructing Second Order Rotatable Designs for any number of factors using any Balanced Incomplete Block Design and if necessary, a suitably chosen set of additional points such as $(o, o \ldots, o)$ or $(\beta, o, o \ldots o) \times 2^1$ type points or $(\beta, \beta, \ldots \beta) \times 2^{t(v)}$ ($2^{t(v)}$ denotes the smallest fractional replicate of 2^v, with levels $+1$ and -1, in which no interaction with less then five factors is conotunded) type points depending on $r=3\lambda$, $r<3\lambda$ and $r>3\lambda$.

New Method of Constructing Second Order Rotatable Designs Through a Pair of Balanced Incomplete Block Designs

A new unified method of constructing Second Order Rotatable Designs by using two suitably chosen Balanced Incomplete Block

Designs, without any additional set of points, can be obtained as follows. The theorem, stated below, forms a generalization of Das and Narasimham [6] method.

Theorm

If $D_1=(v, b_1, r_1, k_1, \lambda_1)$ and

$D_2=(v, b_2, r_2, k_2, \lambda_2)$ are two Balanced Incomplete Block Designs in 'v' treatments with $r_1 \lesseqgtr 3\lambda_1$ and $r_2 \gtreqless 3\lambda_2$ respectively and using multiplication in Das and Narasimham [6] sense, the design points,

$$\alpha-(v, b_1, r_1, k_1, \lambda_1)\times 2^{t(k_1)} \; U \; \beta-(v, b_2, r_2, k_2, \lambda_2)\times 2^{t(k_2)}$$

give a v dimensional Second Order Rotatable Design in

$$N=[b_1\times 2^{t(k_2)}+b_2\times 2^{t(k_2)}] \text{ points,}$$

with
$$\frac{\beta^4}{\alpha^4}=-\frac{(r_1-3\lambda_1)}{(r_2-3\lambda_2)}\times 2\;[t\,(k_1)-t\,(k_2)]$$

PROOF : We have for the above 'N' design points,

$$\sum x_{iu}^4=r_1\; 2^{t(k_1)}\alpha^4+r_2\beta^4\; 2^{t(k_2)}$$

and
$$\sum x_{iu}^2\times x_{ju}^2=\lambda_1\; 2^{t(k_1)}\alpha^4+\lambda_2\; 2^{t(k_2)}\;\beta^4$$

The Rotatability condition,

$$\sum x_{iu}^4=3\sum x_{iu}^2\; x_{ju}^2 \text{ gives}$$

$$r_1\; 2^{t(k_1)}\alpha^4+r_2\beta^4\; 2^{t(k_2)}=3\;\lambda_1\; 2^{t(k1)}\;\alpha^4+3\;\lambda_2\;\beta^4\; 2^{t(k_2)}$$

(*i.e.,*)
$$(r_1-3\lambda_1)\; 2^{t(k_1)}\;\alpha^4+(r_2-3\;\lambda_2)\; 2^{t(k_2)}\;\beta^4=o \qquad ...(3.1)$$

If $r_1-3\lambda_1$ and $r_2=3\lambda_2$, equation (3.1) is satisfied for any α and β. But to get a non-trivial design, at least one of α or β must be different from 0 (zero).

If $r_1\neq 3\;\lambda_1$ and $r_2\neq 3\;\lambda_2$, equation (3.1) gives,

$$\frac{\beta^4}{\alpha^4}=-\frac{(r_1-3\lambda_1)}{(r_2-3\lambda_2)}\times 2\;[^{t(k_1)}-{}^{t(k_2)}] \qquad ...(3.2)$$

Equation (3.2) has a real solution if either $r_1<3\;\lambda_1$ or $r_2<3\lambda_2$, but not both.

COROLLARY : If in D_1, $r_1 \gtrless 3\lambda_1$ and taking a trivial Balanced Incomplete Block Design for D_2 suitably, in the above theorem, this method reduces to Das and Narasimham [6] method.

COROLLARY : Taking $D_1=(v=v,\ b_1=1,\ k=v,\ r_1=1,\ \lambda=1)$ and $D_2=(v=v,\ b_2=v,\ k_2=1, r_2=1,\ \lambda_2=o)$, we get the Central Composite Rotatable Design in

$$N=(2^{t(v)}+2v\} \quad \text{points}$$

with
$$\frac{\beta^4}{\alpha^4}=\frac{2}{1}\times 2\,[^{t(v)-1}]=2^{t(v)}$$

as a particular case.

COROLLARY : If in D_1, $r_1<3\lambda_1$, taking the Balanced Incomplete Block Design $D_2=(v=v,\ b_2=\binom{v}{2},\ k_2=2,\ r_2=(v-1),\ \lambda_2=1)$ with $r_2>3\lambda_2$ and $v>4$ the above theorem gives a new method of constructing Second Order Rotatable Design in

$$N=[b_1\times 2^{t(k_1)}+(2^v)\,2^2] \text{ points}$$

with
$$\frac{\beta^4}{\alpha^4}=-\frac{(r_1-3\lambda_1)}{(v-4)}\times 2[^{t(k_1)-2}$$

COROLLARY : If $D_1=(v=v,\ b_1=1,\ k_1=v,\ r_1=1\ \lambda=1)$ and $D_2=(v=v,\ b_2=\binom{v}{2}\ k_2=2,\ r_2=(v-1),\ \lambda_2=1)$, $v>4$, we get a new series of Second Order Rotatable Designs similar to Central Composite Rotatable Designs in

$$N=[2^{t(v)}+4\,(2^v)] \text{ points}$$

with
$$\frac{\beta^4}{\alpha^4}=\frac{2}{(v-4)}\times 2^{t(v)-2}=\frac{2^{t(v)-1}}{(v-4)}$$

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